Benha University Faculty of Engineering- Shoubra Eng. Mathematics & Physics Department Preparatory Year		Final Term Exam Date: January 12, 2016 Course: Mathematics 1 – A Duration: **3** hours

- The Exam consists of one page
- Answer **All** Questions
- No. of questions: 4
- Total Mark: 100

Question 1

(a)Find **y`** from the following: 12

(i) $y = 2x^3 + 3^{x^2} - 3$ (ii) $y = \cos x^2 . \sec x$

(iii) $y = \ln \sin x + \sin \ln x$ (iv) $\mathrm{y} = \tan^{-3}\mathrm{x} + \log \mathrm{x}$

(v) $2^y + 2^x = \log(x + y)$ (vi) $\mathrm{y} = \mathrm{t} \sin \mathrm{t},\ x = \mathrm{t} + \ln \mathrm{t}$

(b)Find the following limits:

(i) $\lim\limits_{x \to \pi} (\cot x + \csc x)$ (ii) $\lim\limits_{x \to 0} \frac{\ln(1 + x^2)}{3^x - 2^x}$ (iii) $\lim\limits_{x \to 0} \frac{x - \tan x}{x^3 + x^2}$ (iv) $\lim\limits_{x \to \infty} \frac{x^8 - 2^x}{x^8 + 3^x}$ 8

Question 2

(a)Write the Maclurin's series of the function: $f(x) = x\, sin\, x^3$. 4

(b)State and verify the mean value theorem, $f(x) = x - \frac{1}{x}$ in interval [1, 2]. 4

(c)Sketch the curve of each function :

(i) $f(x) = \frac{x}{\sqrt{x^2-1}}$ (ii) $g(x) = \frac{x}{1+x^2}$ 12

(d)Find the integrals:

$\int (x^3 + 3^x)\, dx$, $\int (2^x - 3^x)^2\, dx$, $\int cos^5 2x\, dx$, $\int_0^{\pi} sin^{10} x\, dx$ 10

Question 3

(a)Find the sum of the series : $\sum\limits_{r=1}^{n} \frac{1}{(2r-1)(2r+1)}$ 10

(b)Expand the fraction $\frac{3x+5}{x^3 - x^2 - x + 1}$ into ascending power series of x. 10

(c)Find the real and imaginary part of : sin z and e^{iz} for any complex number z. 5

Question 4	
(a)By Gauss method, solve the system : $x + y + 2z = 7,\quad 2x + y = 8,\quad x + 2y + 2z = 9.$	10
(b)Find the eigenvalues and eigenvector of : $A = \begin{bmatrix} -2 & -2 & -4 \\ 2 & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix}$	10
(c)Find the value of k in : $12x^3 - 8x^2 + kx + 18 = 0$ given that sum of two roots equal zero, then solve it.	5

Good Luck *Dr. Mohamed Eid* *Dr. Fathi Abdsallam*

Answer

Answer of Question 1

(a)(i) $y` = 6x^2 + 3^{x^2}.\ln 3 \,.\, 2x - 0$

(ii) $y` = \cos x^2 \,.\, \sec x \,.\, \tan x - \sin x^2 \,.\, 2x.\sec x$

(iii) $y` = \frac{\cos x}{\sin x} + \cos \ln x \,.\, \frac{1}{x}$

(iv) $y` = -3\, tan^{-4} x.sex^2 x + \frac{1}{\ln 10}.\frac{1}{x}$

(v) $2^y \ln 2 \,.\, y` + 2^x \ln 2 = \frac{1}{\ln 10}.\frac{1+y`}{x+y}$

Then $y`\left(2^y \ln 2 - \frac{1}{\ln 10}.\frac{1}{x+y}\right) = \frac{1}{\ln 10}.\frac{1}{x+y} - 2^x \ln 2$

Then $y` = \frac{\frac{1}{\ln 10}.\frac{1}{x+y} - 2^x \ln 2}{2^y \ln 2 - \frac{1}{\ln 10}.\frac{1}{x+y}}$

(vi) $y` = \frac{\dot{y}}{\dot{x}} = \frac{t \cos t + \sin t}{1 + \frac{1}{t}}$

---*12-Marks* (b)(i) $\lim_{x\to\pi} (\cot x + \csc x) = -\infty + \infty$

Using L'Hopital's rule, we get

$$\lim_{x\to\pi} \left(\frac{\cos x}{\sin x} - \frac{1}{\sin x}\right) = \lim_{x\to\pi} \left(\frac{\cos x - 1}{\sin x}\right) = \lim_{x\to\pi} \left(\frac{-\sin x - 0}{\cos x}\right) = \frac{0}{-1} = 0$$

(ii) $$\lim_{x\to 0} \frac{\ln(1+x^2)}{3^x - 2^x} = \frac{0}{0} = \lim_{x\to 0} \frac{\frac{\ln(1+x^2)}{x^2} . x}{2^x \frac{((3/2)^x - 1)}{x}} = \lim_{x\to 0} \frac{x}{2^x} \frac{\frac{\ln(1+x^2)}{x^2}}{\frac{((3/2)^x - 1)}{x}} = \frac{0}{1} . \frac{1}{\ln(3/2)} = 0$$

(iii) $$\lim_{x\to 0} \frac{x - \tan x}{x^3 + x^2} = \frac{0}{0}$$

Using L'Hopital's rule, we get

$$\lim_{x\to 0} \frac{x - \tan x}{x^3 + x^2} = \lim_{x\to 0} \frac{1 - \sec^2 x}{3x^2 + 2x} = \frac{0}{0} = \lim_{x\to 0} \frac{-2\sec^2 x . \tan x}{6x + 2} = \frac{-2(1)(0)}{0 + 2} = 0$$

(iv) $$\lim_{x\to\infty} \frac{x^8 - 2^x}{x^8 + 3^x} = \frac{\infty - \infty}{\infty + \infty} = \lim_{x\to\infty} \frac{\frac{x^8}{3^x} - (\frac{2}{3})^x}{\frac{x^8}{3^x} + 1} = \frac{0 - 0}{0 + 1} = 0$$

--*8-Marks*

<u>Answer of Question 2</u>

(a)Since $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} ....$

Then $\sin x^3 = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} ....$

Then $f(x) = x \sin x^3 = x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} ....$

--*4-Marks*

(b)The mean value theorem.

We see that $f(x) = x - \frac{1}{x}$ and its derivative $f`(x) = 1 + \frac{1}{x^2}$ are continuous in the given interval [1, 2].

Then $f`(c) = 1 + \frac{1}{c^2} = \frac{f(2)-f(1)}{2-1} = \frac{\frac{3}{2}-0}{1}$. Then $c^2 = 2$ and $c = \pm\sqrt{2}$.

We see that $c = \sqrt{2}$ in [1, 2]. Then, the theorem is verified.-----------*4-Marks*

(c)(i) $f(x) = \frac{x}{\sqrt{x^2-1}}$. Domain f is R – (–1, 1).

- <u>Asymptotic lines</u>

The two lines x = 1 and x = – 1 are vertical asymptotes.

Since $\lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}} = 1, \quad \lim_{x\to-\infty} \frac{x}{\sqrt{x^2-1}} = -1$

Then, the two lines y = 1 and y = – 1 are horizontal asymptotes.

No inclined asymptotes.

- <u>Points of intersection</u>

With x – axis: solve $\frac{x}{\sqrt{x^2-1}} = 0$, we get x = 0 outside the domain, No points

With y – axis: $f(0) = \frac{0}{\sqrt{0-1}} = 0$, but x = 0 outside the domain, No points.

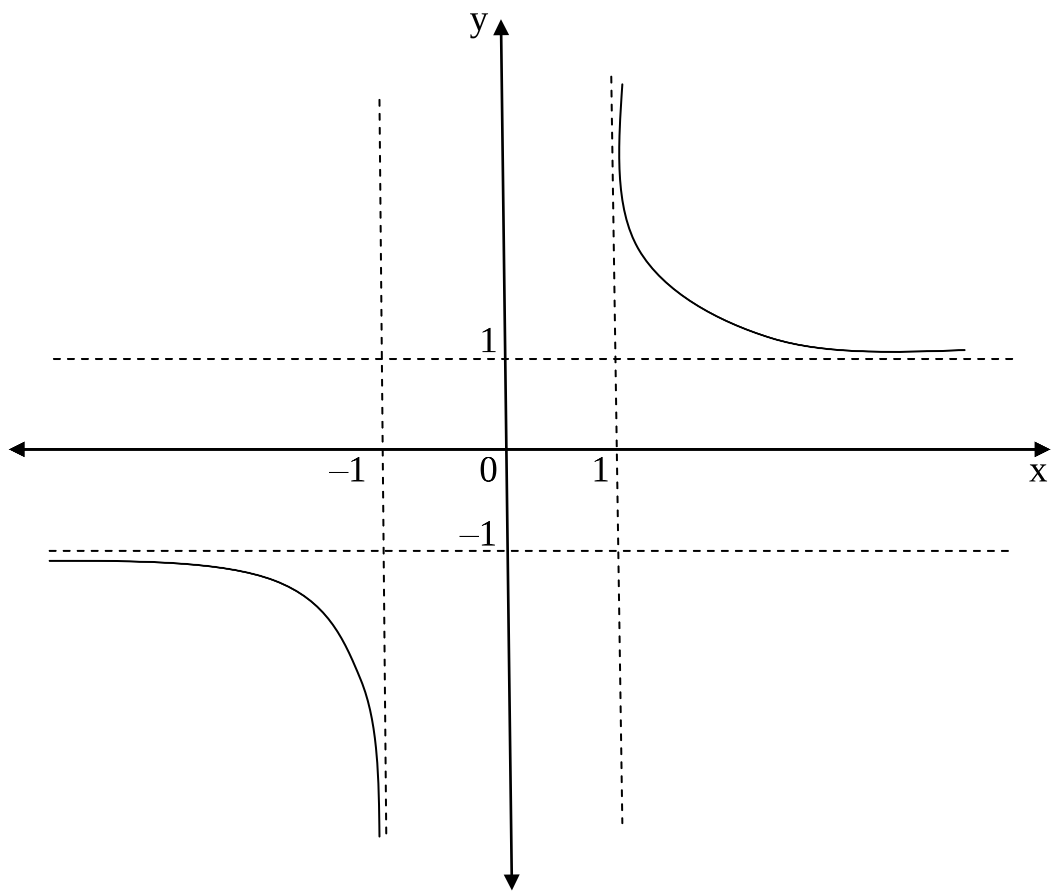
- <u>Extrema</u>

$Since\ \ f`(x) = \frac{\sqrt{x^2-1}.1 - x\frac{2x}{2\sqrt{x^2-1}}}{x^2-1} = \frac{1}{(x^2-1)\sqrt{x^2-1}} \neq 0$. No Extrema.

No Inflection points.

- <u>It is odd function</u> because $f(-x) = f(x)$.

See the figure.



(c)(ii) $g(x) = \frac{x}{1+x^2}$. Domain f is R.

- <u>Asymptotic lines</u>

No vertical asymptotes.

Since $\lim_{x\to\pm\infty} \frac{x}{1+x^2} = 0$

Then, the line y = 0 is horizontal asymptote.

No inclined asymptotes.

- <u>Points of intersection</u>

With x – axis: solve $\frac{x}{1+x^2} = 0$, we get x = 0.

With y – axis: $f(0) = \frac{0}{1+0} = 0$, then y = 0.

- <u>Extrema</u>

$Since\ \ f`(x) = \frac{(1+x^2).1-x.2x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2} = 0$. Then $1 - x^2 = 0$

We get the point (1, 1/2) which is maximum and (–1, –1/2) which is minimum.

- <u>Inflection points</u>

$Since\ \ f``(x) = \frac{(1+x^2)^2.-2x - (1-x^2).2(1+x^2).2x}{(1+x^2)^4} = \frac{2x^3-6x}{(1+x^2)^3} = 0$

Then $2x^3 - 6x = 0$

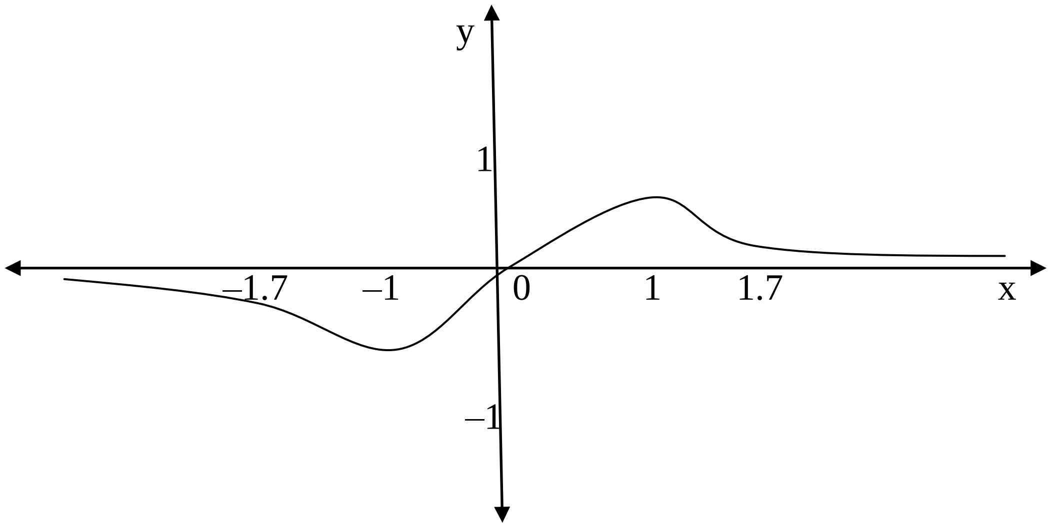
Then $x = 0, \sqrt{3} = 1.7, -\sqrt{3} = -1.7$

We get the inflection points:

$(0,0), \quad \left(\sqrt{3}, \frac{\sqrt{3}}{\sqrt{8}}\right) = (1.7,0.6), \quad \left(-\sqrt{3}, \frac{-\sqrt{3}}{\sqrt{8}}\right) = (-1.7,-0.6)$

- <u>It is odd function</u> because $f(-x) = f(x)$.

See the figure.



---*12-Marks*

(d) $\int (x^3 + 3^x)\, dx = \frac{x^4}{4} + \frac{3^x}{\ln 3} + c$,

$\int (2^x - 3^x)^2\, dx = \int (4^x + 9^x - 2\; 6^x)\, dx = \frac{4^x}{\ln 4} + \frac{9^x}{\ln 9} - 2\,\frac{6^x}{\ln 6} + c$

$\int cos^5 2x\, dx = \int \cos 2x\, [cos^2 2x]^2\, dx = \int \cos 2x\, [1 - sin^2 2x]^2\, dx$

$= \int \cos 2x[1 - 2sin^2 2x + sin^4 2x]\, dx$

$= \int (\cos 2x - 2\cos 2x\, sin^2 2x + \cos 2x\, sin^4 2x)\, dx$

$= \frac{1}{2}\sin 2x - \frac{1}{3} sin^3 2x + \frac{1}{10} sin^5 2x + c$

$\int_0^{\pi} sin^{10} x\, dx = 2\int_0^{\pi/2} sin^{10} x\, dx = 2\,\frac{9.7.5.3.1}{10.8.6.4.2}.\frac{\pi}{2}$

---*10-Marks*

Dr. Mohamed Eid

Answer of Question 3

(a)Find the sum of the series $\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)}$

Solution:

$$u_r = \frac{1}{(2r-1)(2r+1)} = \frac{1}{2}\left[\frac{1}{(2r-1)} + \frac{1}{(2r+1)}\right] = f(r) - f(r+1)$$

$$S_n = f(1) - f(n+1) = \frac{1}{2}\left[\frac{1}{(2-1)} + \frac{1}{(2n+1)}\right] = \frac{n}{2n+1}$$

$$\boxed{\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}}$$

--*10-Marks*

(b)Expand the fraction $\frac{(3x+5)}{x^3 - x^2 - x + 1}$ into ascending power series of x .

Solution:

$$\mathbf{x^3 - x^2 - x + 1 = (x^3 - x^2) - (x-1) = x^2(x-1) - (x-1)}$$
$$\mathbf{= (x-1)(x^2-1) = (x-1)(x-1)(x+1) = (x-1)^2(x+1)}$$

Then $\mathbf{\frac{(3x+5)}{x^3 - x^2 - x + 1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}}$ **(1)**

Then we have

$\mathbf{(3x+5) = A(x-1)^2 + B(x-1)(x+1) + C(x+1)}$ **(2)**

For $x=1 \Rightarrow 2C=8 \Rightarrow \boxed{C=4}$

For $x=-1 \Rightarrow 4A=2 \Rightarrow \boxed{A=1/2}$

And for $x=0 \Rightarrow A-B+C=5 \Rightarrow \boxed{B=-1/2}$

$$\therefore \boxed{\frac{(3x+5)}{x^3-x^2-x+1}=\frac{1/2}{x+1}+\frac{-1/2}{x-1}+\frac{4}{(x-1)^2}}$$

$$\therefore \frac{(3x+5)}{x^3-x^2-x+1}=\frac{1/2}{x+1}+\frac{-1/2}{x-1}+\frac{4}{(x-1)^2}$$

$$=\frac{1}{2}(1+x)^{-1}-\frac{1}{2}(1-x)^{-1}+4(1-x)^{-2}$$

$$\boxed{\frac{(3x+5)}{x^3-x^2-x+1}=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n x^n-\frac{1}{2}\sum_{n=0}^{\infty}x^n+4\sum_{n=0}^{\infty}(n+1)x^n}$$

--*10-Marks*

(c)Find the real and imaginary part to $\sin z$ and e^{iz} for any complex number z

Solution

$$\sin z=\sin(x+iy)=\sin x\cos(iy)+\cos x\sin(iy)$$
$$=\sin x\cosh y+i\cos x\sinh y$$

The real part is $\sin x \cosh y$

The imaginary part is $\cos x \sinh y$

$e^{iz}=e^{i(x+iy)}=e^{-y+ix}=e^{-y}e^{ix}=e^{-y}(\cos x+i\sin x)$

The real part is $e^{-y}\cos x$ and the imaginary part is $e^{-y}\sin x$

--*5-Marks*

Answer of Question 4

(a) Gauss Elimination

$$\left[\begin{array}{ccc|c}1&1&2&7\\2&1&0&8\\1&2&2&9\end{array}\right]\approx\left[\begin{array}{ccc|c}1&1&2&7\\0&-1&-4&-6\\0&1&0&2\end{array}\right]\approx\left[\begin{array}{ccc|c}1&0&-2&1\\0&-1&-4&-6\\0&0&-4&-4\end{array}\right]\approx\left[\begin{array}{ccc|c}1&0&-2&1\\0&-1&-4&-6\\0&0&1&1\end{array}\right]$$

$$\approx\left[\begin{array}{ccc|c}1&0&0&3\\0&1&0&2\\0&0&1&1\end{array}\right] \text{ then } \begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\2\\1\end{bmatrix}$$

---*10-Marks*

(b)Find the eigen values and eigen vectors for the matrix

$$\boldsymbol{A}=\begin{bmatrix}-2&-2&-4\\2&3&2\\3&2&5\end{bmatrix}$$

Solution:

The characteristic polynomial of $\boldsymbol{A}$ is

$$|\boldsymbol{A}-\lambda \boldsymbol{I}|=\begin{vmatrix}-2-\lambda&-2&-4\\2&3-\lambda&2\\3&2&5-\lambda\end{vmatrix}=(\lambda-1)(\lambda-2)(\lambda-3)=0$$

then the characteristic roots are $\lambda=1,2,3$

to determine the eigen vectors we use the equation

$(\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{X}_1'=\mathbf{0}$ where $\boldsymbol{X}_1=(a_1,a_2,a_3)$

at $\lambda=1$ we have

$$\left(A-\lambda I\right)X_1' = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 2 & 2 \\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0$$

this system corresponds to

$a_1 + 2a_3 = 0,\ a_2 - a_3 = 0$ let $a_3 = t \neq 0$

Then $X_1 = (-2t, t, t)$ if $t = 1$ then $X_1 = (-2, 1, 1)$

at $\lambda = 2$ let $X_2 = (b_1, b_2, b_3)$ we and

$$\left(A-\lambda I\right)X_2' = \begin{bmatrix} -4 & -2 & -4 \\ 2 & 1 & 2 \\ 3 & 2 & 3 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = 0$$

Then $b_1 + b_3 = 0,\ b_2 = 0$ and $X_2 = (-1, 0, 1)$

When $\lambda = 3$ let $X_3 = (c_1, c_2, c_3)$ and

$$\left(A-\lambda I\right)X_3' = \begin{bmatrix} -5 & -2 & -4 \\ 2 & 0 & 2 \\ 3 & 2 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 3 & 2 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$$

then $c_1 + c_3 = 0,\ 2c_2 - c_3 = 0$ $X_3 = (-2, 1, 2)$

--*10-Marks*

(c)Solve $12x^3 - 8x^2 + kx + 18 = 0$ and find the value of k given that sum of two roots equal zero.

Solution

Let the roots are a, b *and* c

The sum of two roots equal zero we put $a+b=0$

The relation between the roots and coefficients

$a+b+c=\frac{8}{12}=\frac{2}{3}$ and since $a+b=0$ then $c=\frac{2}{3}$

$abc=\frac{-18}{12}=\frac{-3}{2}$ and since $c=\frac{2}{3}$ then $ab=\frac{-9}{4}$

a and c are roots for $x^2-\frac{9}{4}=0$ then $a=\frac{3}{2}$ and $c=\frac{-3}{2}$

and $k=12(ab+ac+bc)$

---*5-Marks*

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